Large cardinal axioms in category theory II

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Before starting on Part II...

Answer to Martin's Question

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Can there be a dense set of objects in **Top***?*

The answer is no. In fact there can't even be a colimit-dense one.

For any regular cardinal κ, consider the co-bounded topology on κ: a set $X \subseteq \kappa$ is open if and only if $\kappa \setminus X$ is bounded in κ. Note that the induced topology on any bounded subset of κ is thus the discrete topology. Therefore, this space cannot be a colimit of any set of spaces of size less than κ: each of them would have bounded and hence discrete image, so the colimit topology would be discrete; but there are non-open sets in this space.

Part II: Accessible categories and Vopěnka's Principle

Functors

The notion of a functor is the natural notion of a homomorphism for categories. A functor between two categories takes objects to objects, morphisms to morphisms, and respects composition and identities.

Examples

- The fundamental group functor $\pi_1 \colon \mathbf{Top} \to \mathbf{Gp}$ in algebraic topology.
- The forgetful functor $U \colon \mathbf{Gp} \to \mathbf{Set}$, sending any group to its underlying set.
- The free group functor $F \colon \mathbf{Set} \to \mathbf{Gp}$, sending any set to the free group generated by that set.
- The abelianisation functor $Ab \colon \mathbf{Gp} \to \mathbf{AbGp}$, sending any group to its abelianisation.
- Instead of viewing a diagram as an ad-hoc collection of objects and morphisms, we can view it as the image of a functor from some *index category*.

Poset categories

Any partially ordered set can be viewed as a category, where the elements of the set are the objects, and there's at most one morphism between any x and y, specifically,

$$|\operatorname{Hom}(x,y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

(Note that the identity morphism will be the unique morphism from x to x.)

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For λ a regular cardinal, a poset is λ-directed if every subset of cardinality less than λ has an upper bound.

Eg: the usual notion of directed is $\aleph_0$-directed in this notation.

A λ-directed diagram is one whose index category (i.e. the domain of the functor) is a λ-directed poset.

λ-presentable objects

Definition

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i.e. for every λ-directed diagram $\mathcal{D}$ in $\mathcal{C}$ with a colimit C, the set $\mathrm{Hom}_{\mathcal{C}}(P, C)$ is built up as the colimit of the sets $\mathrm{Hom}_{\mathcal{C}}(P, D_i)$ (for D_i in $\mathcal{D}$) along maps obtained by composing with the morphisms in $\mathcal{D}$.

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i.e. for every morphism f from P to the colimit C of a λ-directed diagram $\mathcal{D}$, f factors through $\mathcal{D}$, uniquely up to the expected identifications via the morphisms in $\mathcal{D}$.

λ-presentability examples

In **Set**, an object is λ-presentable iff it has cardinality less than λ.

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In **Gp**, $\aleph_0$ presentability agrees with the usual notion of finite presentability. Likewise, a group is λ-presentable if it is given by fewer than λ many generators and relations.

Accessible categories

A category $\mathcal{K}$ is λ-accessible if

- $\mathcal{K}$ has λ-directed colimits, and
- there is a *set* $\mathcal{A}$ of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathcal{A}$.

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A category $\mathcal{K}$ is locally λ-presentable if it is λ-accessible and it *cocomplete*: it has colimits for all small (i.e. set-sized) diagrams.

Examples

Locally presentable categories

- **Set**, **Gp**
- **Gra**, the category of directed graphs and their homomorphisms.
- **Ban**, the category of complex Banach spaces and linear contractions.

Accessible but not locally presentable categories

- The category of sets with only injections as morphisms.
- **Hil**, the category of Hilbert spaces as a full subcategory of **Ban** (i.e. with linear contractions as the morphisms).
- The category of models for some theory, with elementary embeddings as the morphisms.

Neither

- **Top** (see the answer to Martin's question).

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Proof outline - details on the board.

For $\mathcal{A}$ the set of λ-presentable objects as per the definition of λ-accessibility, there are no more than

$$\sum_{A \in \mathcal{A}} |\operatorname{Hom}(A, A)|$$

isomorphism types of λ-presentable objects in $\mathcal{K}$. Indeed, for every λ-presentable object K, there are an $A \in \mathcal{A}$ and morphisms $f: A \to K$ and $g: K \to A$ such that K is identified up to isomorphism by $g \circ f$. □

Simplification II

For any λ-accessible category $\mathcal{K}$, let $\mathbf{Pres}_\lambda\mathcal{K}$ denote the set of (chosen representatives of the isomorphism classes of) all λ-presentable objects in $\mathcal{K}$.

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Theorem

In any λ-accessible category $\mathcal{K}$, $\mathbf{Pres}_\lambda\mathcal{K}$ is dense, and the canonical diagrams have λ-directed cofinal subdiagrams.

Proof outline.

Use the defining colimits of λ-accessibility, and λ-presentability. □

A theorem

Theorem (Rosický, Trnková & Adámek, 1990)

Assuming Vopěnka's Principle, for each full embedding $F : \mathcal{A} \to \mathcal{K}$ with $\mathcal{K}$ an accessible category, there is a regular cardinal λ such that F preserves λ-directed colimits.

Vopěnka's Principle

Vopěnka's Principle, accessible categories formulation

No accessible category admits a proper class of objects with the only morphisms amongst them being the identities (i.e. there is no *rigid* class of objects).

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Vopěnka's Principle, graphs formulation

For any proper class $\mathcal{C}$ of graphs, there are distinct graphs G_0 and G_1 in $\mathcal{C}$ such that there exists a graph homomorphism from G_0 to G_1.

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Vopěnka's Principle, first order structres formulation

For any signature $\mathbf{\Sigma}$, and any proper class $\mathcal{C}$ of $\mathbf{\Sigma}$-structures, there are distinct structures A and B in $\mathcal{C}$ such that there exists a homomorphism from A to B.

Vopěnka's Principle, elementary embeddings formulation

For any signature $\mathbf{\Sigma}$, and any proper class $\mathcal{C}$ of $\mathbf{\Sigma}$-structures, there are distinct structures A and B in $\mathcal{C}$ such that there exists an elementary embedding from A to B.

An important observation

Bagaria, Casacuberta, Mathias and Rosický: VP for classes defined by formulae of a given quantifier complexity is strictly weaker than full VP, so many specific applications of VP can be obtained from weaker assumptions.

A convenient sledgehammer

Theorem (Rosický)

The accessible categories are precisely those equivalent to the categories of models of **basic** *theories (possibly in infinitary logic). In particular, each one can be fully embedded in* $\mathbf{Str}\,\Sigma$ *for suitable* Σ.

Stratifying the theorem

Theorem (Bagaria & B-T)

Suppose that $\mathcal{K}$ is a full subcategory of $\mathbf{Str}\,\boldsymbol{\Sigma}$ for some signature $\boldsymbol{\Sigma}$. Let $F : \mathcal{A} \to \mathcal{K}$ be any Σ_n-definable full embedding with Σ_n-definable domain category $\mathcal{A}$, for some $n > 0$. If there exists a $C^{(n)}$-extendible cardinal greater than

- *the rank of $\boldsymbol{\Sigma}$,*
- *the arity of each function or relation symbol in $\boldsymbol{\Sigma}$, and*
- *the ranks of the parameters used in some Σ_n definitions of F and $\mathcal{A}$ and in some definition of $\mathcal{K}$,*

then there exists a regular cardinal λ such that F preserves λ-directed colimits.

Using elementary embeddings of the universe in category theory

Basic idea

If all the categories, functors, etc that you care about are definable, and j is an elementary embedding of the whole universe into (a reasonable approximation of) itself fixing the parameters in the definitions, then j commutes with everything.

$C^{(n)}$ cardinals

We denote by $C^{(n)}$ the class of cardinals κ such that $V_\kappa \prec_{\Sigma_n} V$, that is, for every Σ_n formula $\varphi(x)$ and set $x_0 \in V_\kappa$,

$$\langle V_\kappa, \in \rangle \models \varphi(x_0) \quad \text{if and only if} \quad \langle V, \in \rangle \models \varphi(x_0).$$

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By the reflection theorem, $C^{(n)}$ is unbounded.

$C^{(n)}$-extendible cardinals

Definition

A cardinal κ is $C^{(n)}$-*extendible* if for every $\lambda > \kappa$ there is a cardinal $\mu > \lambda$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that

1. $\kappa = \mathrm{crit}(j)$, i.e., κ is the least ordinal such that $j(\kappa) \neq \kappa$,
2. $j(\kappa) > \lambda$, and
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κ is $C^{(n)+}$-*extendible* if moreover, for every $\lambda > \kappa$ in $C^{(n)}$, there is a $\mu > \lambda$ in $C^{(n)}$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that (1), (2) and (3) hold.

Relationships of large cardinals

Theorem (Tsaprounis)

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A cardinal κ is $C^{(n)}$-extendible if and only if it is $C^{(n)+}$-extendible.

Theorem (Bagaria, Casacuberta, Mathias & Rosický)

Vopěnka's Principle is equivalent to the existence of a proper class of $C^{(n)+}$-extendible cardinals for every n. Morever, the existence of a $C^{(n)+}$-extendible κ corresponds exactly to Vopěnka's Principle for classes that are Σ_{n+2}-definable with parameters from V_κ.

The main theorem again

Theorem (Bagaria & B-T)

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1. *the rank of $\boldsymbol{\Sigma}$,*
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3. *the ranks of the parameters used in some Σ_n definitions of F and $\mathcal{A}$ and in some definition of $\mathcal{K}$,*

then there exists a regular cardinal λ such that F preserves λ-directed colimits.

Proof

Want to show $F : \mathcal{A} \to \mathcal{K}$ preserves λ-directed colimits.

Sufficient:

$$i \circ F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma} \text{ preserves } \lambda\text{-directed colimits,}$$

where $i : \mathcal{K} \to \mathbf{Str}\,\mathbf{\Sigma}$ is the inclusion functor (and this notional inclusion doesn't change the quantifier complexity). So WLOG assume $F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma}$.

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Note: $\mathbf{Str}\,\mathbf{\Sigma}$ has all λ-directed colimits, for λ greater than the arities of the symbols in $\mathbf{\Sigma}$ (i.e. cardinals as per (ii)).

Let β be sufficiently large as per (i), (ii) and (iii).

Proof

Want to show $F : \mathcal{A} \to \mathbf{Str}\,\boldsymbol{\Sigma}$ preserves λ-directed colimits.

Consider the following category $\mathcal{C}$:

Objects: $\mathbf{Str}\,\boldsymbol{\Sigma}$ morphisms $a : \bar{A} \to F(A)$ such that for some $\lambda > \beta$ and some λ-directed diagram $\mathcal{D}$ in $\mathcal{A}$,

- A is the colimit of $\mathcal{D}$ in $\mathcal{A}$,
- $\bar{A}$ is the colimit of $F\mathcal{D}$ in $\mathbf{Str}\,\boldsymbol{\Sigma}$, and
- a is the morphism induced by the image under F of the $\mathcal{A}$-colimit cocone from $\mathcal{D}$ to A.

Morphisms: From a to b: pairs $\langle g, h \rangle$ of $\mathbf{Str}\,\boldsymbol{\Sigma}$ morphisms such that

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\ a\ } & F(A) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \bar{B} & \xrightarrow[\ b\ ]{} & F(B) \end{array}$$

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Let $\mathcal{C}^*$ be the full subcategory of $\mathcal{C}$ of those a which are *not* isomorphisms.

If the theorem fails, then $\mathcal{C}^*$ is not essentially small.

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$a \in \mathrm{Obj}(\mathcal{C}^*)$ iff

$$\begin{aligned}\exists\lambda\exists\mathcal{D}\exists\langle\bar{A},\bar{\eta}\rangle\exists\langle A,\eta\rangle(&\lambda \text{ is a regular cardinal} \wedge \mathcal{D} \text{ is a diagram in } \mathcal{A} \wedge\\ &\mathcal{D} \text{ is } \lambda\text{-directed} \wedge\\ &\langle\bar{A},\bar{\eta}\rangle = \mathrm{Colim}_{\mathbf{Str\,\Sigma}}(F\mathcal{D}) \wedge \langle A,\eta\rangle = \mathrm{Colim}_{\mathcal{A}}(\mathcal{D}) \wedge\\ &a : \bar{A} \to F(A) \text{ is the induced homomorphism} \wedge\\ &a \text{ is not an isomorphism}).\end{aligned}$$

The universal property of colimits makes the middle line Π_{n+1}.

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Let κ be a $C^{(n)+}$-extendible cardinal greater than β.

Let a be an object of $\mathcal{C}^*$ of rank $> \kappa$, arising from a λ_a-directed diagram $\mathcal{D}_a$ for some $\lambda_a > \kappa$.

Let $\lambda \in C^{(n)}$ be greater than the ranks of $a, \mathcal{D}_a, F\mathcal{D}_a$, and the corresponding cocones $\langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$.

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$$V_\mu \vDash \lambda_a, \mathcal{D}_a, \langle \bar{A}, \bar{\eta} \rangle_a \text{ and } \langle A, \eta \rangle_a \text{ witness that } a \in \mathrm{Obj}(\mathcal{C}^*).$$

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$$V_\mu \vDash \lambda_a, \mathcal{D}_a, \langle \bar{A}, \bar{\eta} \rangle_a \text{ and } \langle A, \eta \rangle_a \text{ witness that } a \in \mathrm{Obj}(\mathcal{C}^*).$$

Henceforth work in V_μ.

Proof

Want to show $F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma}$ preserves λ-directed colimits.
$\mathcal{C}$: cat. of λ-directed colimit morphisms $a : \bar{A} \to F(A)$
$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is Σ_{n+2}-definable
Want to show $\mathcal{C}^*$ is essentially small
If not take κ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ "bigger", $\lambda \in C^{(n)}$ yet bigger,
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Note that because $\kappa > \beta$, the definition of F is unaffected by j, so j commutes with F.

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Since j is elementary, we have a morphism in ${\mathcal{C}^*}^{V_\mu}$

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\quad a \quad} & F(A) \\ {\scriptstyle j\restriction\bar{A}}\downarrow & & \downarrow{\scriptstyle j\restriction F(A)} \\ j(\bar{A}) & \xrightarrow[\quad j(a) \quad]{} & j(F(A)). \end{array}$$

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Now, $\mathcal{D}_a$ is λ_a-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed.

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$$\begin{CD} \bar{A} @>{a}>> F(A) \\ @V{j\restriction\bar{A}}VV @VV{j\restriction F(A)}V \\ j(\bar{A}) @>>{j(a)}> j(F(A)). \end{CD}$$

Now, $\mathcal{D}_a$ is λ_a-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed. Since $j(\lambda_a) > j(\kappa) > \lambda > |\mathcal{D}_a|$, $j``F\mathcal{D}_a$ has an upper bound $F(d_0)$ in $Fj(\mathcal{D}_a)$.

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