Large cardinal axioms in category theory II

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Winter School 2020

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Before starting on Part II...

Answer to Martin's Question

Martin asked:

Question

Can there be a dense set of objects in **Top**?

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The answer is no. In fact there can't even be a colimit-dense one.

For any regular cardinal κ , consider the co-bounded topology on κ : a set $X \subseteq \kappa$ is open if and only if $\kappa \setminus X$ is bounded in κ . Note that the induced topology on any bounded subset of κ is thus the discrete topology. Therefore, this space cannot be a colimit of any set of spaces of size less than κ : each of them would have bounded and hence discrete image, so the colimit topology would be discrete; but there are non-open sets in this space.

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Part II: Accessible categories and Vopěnka's Principle

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Functors

The notion of a functor is the natural notion of a homomorphism for categories. A functor between two categories takes objects to objects, morphisms to morphisms, and respects composition and identities.

Examples

- The fundamental group functor π_1 : **Top** \rightarrow **Gp** in algebraic topology.
- The forgetful functor $U \colon \mathbf{Gp} \to \mathbf{Set}$, sending any group to its underlying set.
- The free group functor $F : \mathbf{Set} \to \mathbf{Gp}$, sending any set to the free group generated by that set.
- The abelianisation functor $Ab: \mathbf{Gp} \to \mathbf{AbGp}$, sending any group to its abelianisation.
- Instead of viewing a diagram as an ad-hoc collection of objects and morphisms, we can view it as the image of a functor from some *index category*.

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Poset categories

Any partially ordered set can be viewed as a category, where the elements of the set are the objects, and there's at most one morphism between any x and y, specifically,

$$|\operatorname{\mathsf{Hom}}(x,y)| = egin{cases} 1 & ext{if } x \leq y \ 0 & ext{otherwise.} \end{cases}$$

(Note that the identity morphism will be the unique morphism from x to x.)

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For λ a regular cardinal, a poset is λ -directed if every subset of cardinality less than λ has an upper bound.

Eg: the usual notion of directed is \aleph_0 -directed in this notation.

A λ -directed diagram is one whose index category (i.e. the domain of the functor) is a λ -directed poset.

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λ -presentable objects

Definition

An object *P* in a category *C* is λ -presentable if the functor $\text{Hom}_{\mathcal{C}}(P, -) \colon \mathcal{C} \to \mathbf{Set}$ preserves λ -direct colimits.

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i.e. for every λ -directed diagram \mathcal{D} in \mathcal{C} with a colimit C, the set $\text{Hom}_{\mathcal{C}}(P, C)$ is built up as the colimit of the sets $\text{Hom}_{\mathcal{C}}(P, D_i)$ (for D_i in \mathcal{D}) along maps obtained by composing with the morphisms in \mathcal{D} .

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i.e. for every morphism f from P to the colimit C of a λ -directed diagram \mathcal{D} , f factors through \mathcal{D} , uniquely up to the expected identifications via the morphisms in \mathcal{D} .

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λ -presentability examples

In **Set**, an object is λ -presentable iff it has cardinality less than λ .

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λ -presentability examples

In **Set**, an object is λ -presentable iff it has cardinality less than λ .

In **Gp**, \aleph_0 presentability agrees with the usual notion of finite presentability. Likewise, a group is λ -presentable if it is given by fewer than λ many generators and relations.

(a)

Accessible categories

A category ${\mathcal K}$ is $\lambda\text{-accessible}$ if

- ${\cal K}$ has $\lambda\text{-directed colimits, and}$
- there is a set A of λ-presentable objects such that every object is a λ-directed colimit of objects from A.

A category is accessible if it is λ -accessible for some λ .

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A category \mathcal{K} is locally λ -presentable if it is λ -accessible and it *cocomplete*: it has colimits for all small (i.e. set-sized) diagrams.

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Examples

Locally presentable categories

- Set, Gp
- Gra, the category of directed graphs and their homomorphisms.
- Ban, the category of complex Banach spaces and linear contractions.

Accessible but not locally presentable categories

- The category of sets with only injections as morphisms.
- **Hil**, the category of Hilbert spaces as a full subcategory of **Ban** (i.e. with linear contractions as the morphisms).
- The category of models for some theory, with elementary embeddings as the morphisms.

Neither

• **Top** (see the answer to Martin's question).

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Simplification I

Theorem

In a λ -accessible category K, there are only set-many isomorphism types of λ -presentable objects.

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Proof outline - details on the board.

For ${\cal A}$ the set of $\lambda\text{-presentable}$ objects as per the definition of $\lambda\text{-accessibility},$ there are no more than

$$\sum_{A\in\mathcal{A}}|\operatorname{Hom}(A,A)|$$

isomorphism types of λ -presentable objects in \mathcal{K} . Indeed, for every λ -presentable object K, there are an $A \in \mathcal{A}$ and morphisms $f : A \to K$ and $g : K \to A$ such that K is identified up to isomorphism by $g \circ f$.

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Simplification II

For any λ -accessible category \mathcal{K} , let $\operatorname{Pres}_{\lambda}\mathcal{K}$ denote the set of (chosen representatives of the isomorphism classes of) all λ -presentable objects in \mathcal{K} .

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Simplification II

For any λ -accessible category \mathcal{K} , let $\operatorname{Pres}_{\lambda}\mathcal{K}$ denote the set of (chosen representatives of the isomorphism classes of) all λ -presentable objects in \mathcal{K} .

Theorem

In any λ -accessible category \mathcal{K} , **Pres**_{λ} \mathcal{K} is dense, and the canonical diagrams have λ -directed cofinal subdiagrams.

Proof outline.

Use the defining colimits of λ -accessibility, and λ -presentability.

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A theorem

Theorem (Rosický, Trnková & Adámek, 1990)

Assuming Vopěnka's Principle, for each full embedding $F : A \to K$ with K an accessible category, there is a regular cardinal λ such that F preserves λ -directed colimits.

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Vopěnka's Principle

Vopěnka's Principle, accessible categories formulation

No accessible category admits a proper class of objects with the only morphisms amongst them being the identities (i.e. there is no *rigid* class of objects).

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Equivalently, by clever coding:

Vopěnka's Principle, graphs formulation

For any proper class C of graphs, there are distinct graphs G_0 and G_1 in C such that there exists a graph homomorphism from G_0 to G_1 .

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For any proper class C of graphs, there are distinct graphs G_0 and G_1 in C such that there exists a graph homomorphism from G_0 to G_1 .

Vopěnka's Principle, first order structres formulation

For any signature Σ , and any proper class C of Σ -structures, there are distinct structures A and B in C such that there exists a homomorphism from A to B.

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Vopěnka's Principle, elementary embeddings formulation

For any signature Σ , and any proper class C of Σ -structures, there are distinct structures A and B in C such that there exists an elementary embedding from A to B.

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An important observation

Bagaria, Casacuberta, Mathias and Rosický: VP for classes defined by formulae of a given quantifier complexity is strictly weaker than full VP, so many specific applications of VP can be obtained from weaker assumptions.

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A convenient sledgehammer

Theorem (Rosický)

The accessible categories are precisely those equivalent to the categories of models of **basic** theories (possibly in infinitary logic). In particular, each one can be fully embedded in **Str** Σ for suitable Σ .

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Stratifying the theorem

Theorem (Bagaria & B-T)

Suppose that \mathcal{K} is a full subcategory of $\operatorname{Str} \Sigma$ for some signature Σ . Let $F : \mathcal{A} \to \mathcal{K}$ be any Σ_n -definable full embedding with Σ_n -definable domain category \mathcal{A} , for some n > 0. If there exists a $C^{(n)}$ -extendible cardinal greater than

- the rank of **Σ**,
- $\bullet\,$ the arity of each function or relation symbol in $\pmb{\Sigma},$ and
- the ranks of the parameters used in some Σ_n definitions of F and A and in some definition of \mathcal{K} ,

then there exists a regular cardinal λ such that F preserves λ -directed colimits.

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Using elementary embeddings of the universe in category theory

Basic idea

If all the categories, functors, etc that you care about are definable, and j is an elementary embedding of the whole universe into (a reasonable approximation of) itself fixing the parameters in the definitions, then j commutes with everything.

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$C^{(n)}$ cardinals

We denote by $C^{(n)}$ the class of cardinals κ such that $V_{\kappa} \prec_{\Sigma_n} V$, that is, for every Σ_n formula $\varphi(x)$ and set $x_0 \in V_{\kappa}$,

$$\langle V_{\kappa}, \in \rangle \vDash \varphi(x_0)$$
 if and only if $\langle V, \in \rangle \vDash \varphi(x_0)$.

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By the reflection theorem, $C^{(n)}$ is unbounded.

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$C^{(n)}$ -extendible cardinals

Definition

A cardinal κ is $C^{(n)}$ -extendible if for every $\lambda > \kappa$ there is a cardinal $\mu > \lambda$ and an elementary embedding $j: V_{\lambda} \to V_{\mu}$ such that

- $\kappa = \operatorname{crit}(j)$, i.e., κ is the least ordinal such that $j(\kappa) \neq \kappa$,
- (a) $j(\kappa) > \lambda$, and
- $j(\kappa) \in C^{(n)}$.

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2)
$$j(\kappa) > \lambda$$
, and

$$i(\kappa) \in C^{(n)}.$$

 κ is $C^{(n)+}$ -extendible if moreover, for every $\lambda > \kappa$ in $C^{(n)}$, there is a $\mu > \lambda$ in $C^{(n)}$ and an elementary embedding $j: V_{\lambda} \to V_{\mu}$ such that (1), (2) and (3) hold.

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Relationships of large cardinals

Theorem (Tsaprounis)

A cardinal κ is $C^{(n)}$ -extendible if and only if it is $C^{(n)+}$ -extendible.

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A cardinal κ is $C^{(n)}$ -extendible if and only if it is $C^{(n)+}$ -extendible.

Theorem (Bagaria, Casacuberta, Mathias & Rosický)

Vopěnka's Principle is equivalent to the existence of a proper class of $C^{(n)+}$ -extendible cardinals for every n. Morever, the existence of a $C^{(n)+}$ -extendible κ corresponds exactly to Vopěnka's Principle for classes that are Σ_{n+2} -definable with parameters from V_{κ} .

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The main theorem again

Theorem (Bagaria & B-T)

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- the rank of Σ ,
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then there exists a regular cardinal λ such that F preserves λ -directed colimits.

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Proof

Want to show $F : \mathcal{A} \to \mathcal{K}$ preserves λ -directed colimits.

Sufficient:

 $i \circ F : \mathcal{A} \to \mathsf{Str}\, \Sigma$ preserves λ -directed colimits,

where $i : \mathcal{K} \to \mathbf{Str} \Sigma$ is the inclusion functor (and this notional inclusion doesn't change the quantifier complexity). So WLOG assume $F : \mathcal{A} \to \mathbf{Str} \Sigma$.

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Note: Str Σ has all λ -directed colimits, for λ greater than the arities of the symbols in Σ (i.e. cardinals as per (ii)).

Let β be sufficiently large as per (i), (ii) and (iii).

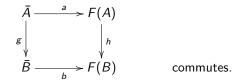
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Consider the following category \mathcal{C} :

- Objects: Str Σ morphisms $a : \overline{A} \to F(A)$ such that for some $\lambda > \beta$ and some λ -directed diagram \mathcal{D} in \mathcal{A} ,
 - A is the colimit of \mathcal{D} in \mathcal{A} ,
 - \overline{A} is the colimit of $F\mathcal{D}$ in **Str** Σ , and
 - *a* is the morphism induced by the image under *F* of the *A*-colimit cocone from *D* to *A*.

Morphisms: From a to b: pairs $\langle g, h \rangle$ of Str Σ morphisms such that



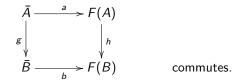
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Let C^* be the full subcategory of C of those *a* which are *not* isomorphisms.

If the theorem fails, then \mathcal{C}^* is not essentially small.

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 $Obj(\mathcal{C}^*)$ is Σ_{n+2} -definable over the language of set theory (extended with \mathcal{P}_{β}):

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 $Obj(\mathcal{C}^*)$ is Σ_{n+2} -definable over the language of set theory (extended with \mathcal{P}_{β}): $a \in Obj(\mathcal{C}^*)$ iff

 $\exists \lambda \exists \mathcal{D} \exists \langle \bar{A}, \bar{\eta} \rangle \exists \langle A, \eta \rangle (\lambda \text{ is a regular cardinal } \land \mathcal{D} \text{ is a diagram in } \mathcal{A} \land \\ \mathcal{D} \text{ is } \lambda \text{-directed } \land \\ \langle \bar{A}, \bar{\eta} \rangle = \text{Colim}_{\mathsf{Str}\,\mathsf{\Sigma}}(F\mathcal{D}) \land \langle A, \eta \rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D}) \land \\ a : \bar{A} \to F(A) \text{ is the induced homomorphism } \land \\ a \text{ is not an isomorphism}).$

The universal property of colimits makes the middle line Π_{n+1} .

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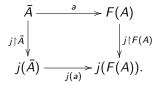
Note that because $\kappa > \beta$, the definition of *F* is unaffected by *j*, so *j* commutes with *F*.

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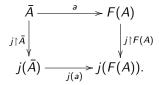
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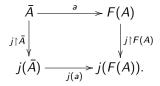
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